

C-AD MAC, 25 November 2013
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Dynamical effects of synchrotron radiation in FFAG eRHIC arcs.

Spin diffusion.

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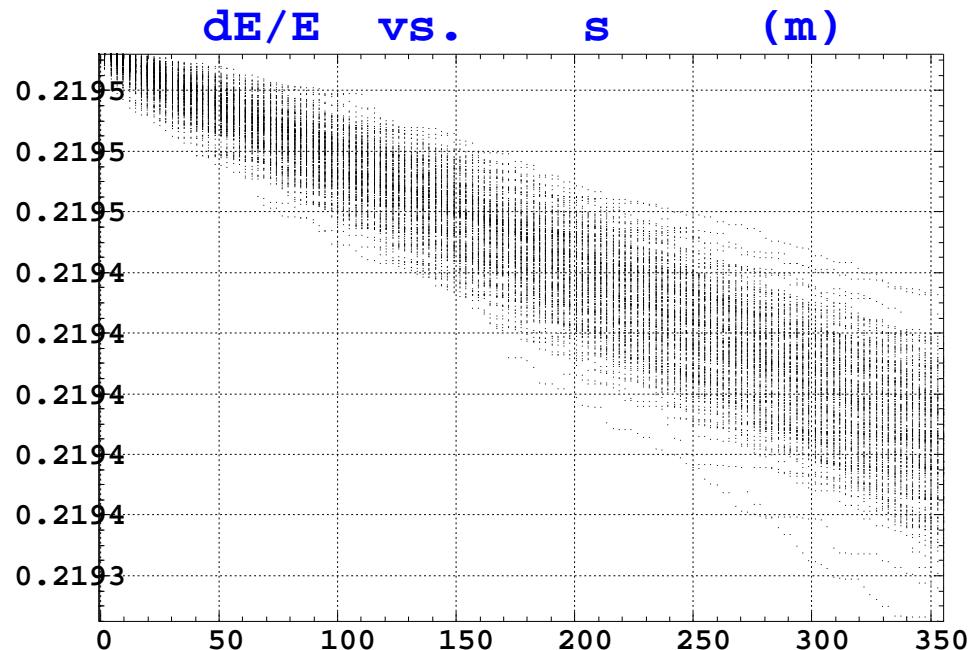
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1 Reminder, dynamical effects of synchrotron radiation

1.1 Particle dynamics

- Electrons circulating in eRHIC arcs loose energy by synchrotron radiation (SR)

Energy loss at top energy
in an eRHIC FFAG arc, $\overline{\Delta E}/E$

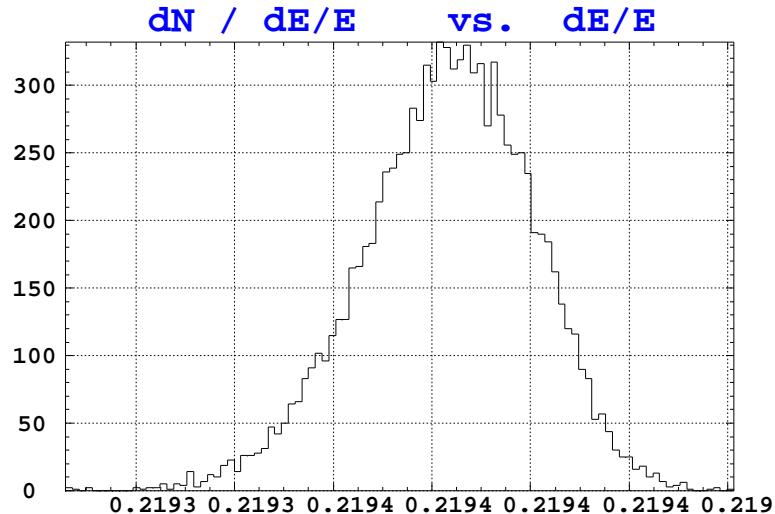


Over a trajectory arc $\Delta\theta$,
with constant curvature $1/\rho$:

$$\frac{\overline{\Delta E}}{E} = 1.879 \times 10^{-15} \frac{\gamma^3}{\rho} \Delta\theta$$

- SR is a stochastic process, photons fluctuate in number and energy, thus inducing

Energy spread, σ_E/E



- energy spread,

$$\frac{\sigma_E}{E} = 3.794 \times 10^{-14} \frac{\gamma^{5/2}}{\rho} \sqrt{\Delta\theta}$$

- and bunch lengthening^(*),

$$\sigma_l = \left(\frac{\sigma_E}{E} \right) \left[\frac{1}{L_{\text{bend}}} \int_0^{L_{\text{bend}}} (D_x(s)T_{51}(s_f \leftarrow s) + D'_x(s)T_{52}(s_f \leftarrow s) - T_{56})^2 ds \right]^{1/2}$$

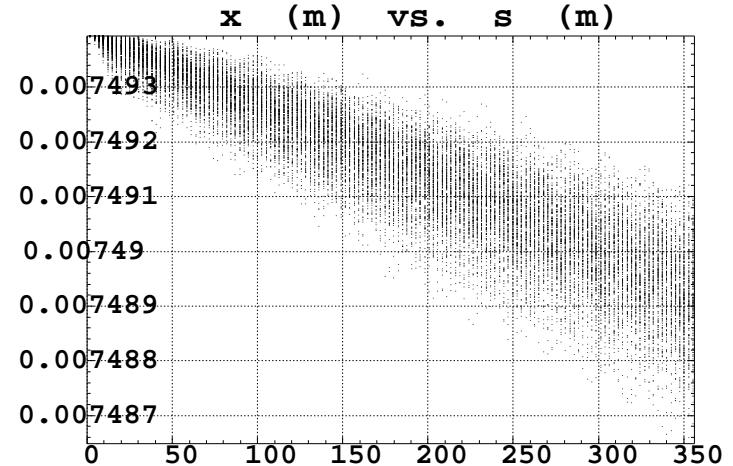
^(*) The linear model is installed in a computer code of ours, BETA/Saclay.

- The energy loss causes a displacement of the beam centroid^(*).

Over a distance $[s_i, s_f]$:

$$\left[\begin{array}{c} \overline{x(s_f)} \\ \overline{x'(s_f)} \end{array} \right] = T(s_f \leftarrow s_i) \times \left[\begin{array}{c} \overline{x(s_i)} \\ \overline{x'(s_i)} \end{array} \right] + \frac{\sigma_E}{E} \left\{ \begin{array}{c} \langle U \rangle \\ \langle V \rangle \end{array} \right\}$$

Inward spiraling of the electron beam in the high energy arc

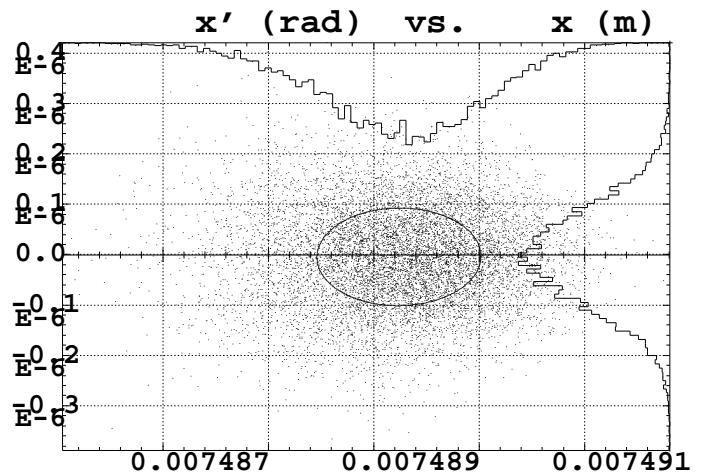


Horizontal phase space at the downstream end of an high energy arc, all initial emittances were taken zero.

- and horizontal emittance growth^(*),

$$\Delta\sigma(s_f) = T(s_f \leftarrow s_i) \times$$

$$\left(\frac{\sigma_E}{E} \right)^2 \left[\begin{array}{cc} \langle U^2 \rangle & \langle UV \rangle \\ \langle UV \rangle & \langle V^2 \rangle \end{array} \right] \times \tilde{T}(s_f \leftarrow s_i)$$



^(*) The linear model is installed in a computer code of ours, BETA.

1.2 Spin dynamics

- Spin transport has to undergo similar treatment as particle motion, considering the building up of averages and second momenta upon random SR. However some simple considerations can serve as a guidance, as follows

- **Spin rotation**

$$\phi = G\gamma\theta$$

wherein

$G = 1.16 \cdot 10^{-3}$ is the anomalous gyromagnetic factor (often noted “a” for electrons),

γ is the Lorentz relativistic factor

θ is the particle deflection angle.

- **Spin angle spreading over an arc (138 cells)**

From $\phi = G\gamma\theta$ it results that a change in energy upon emission of a photon will cause a change in spin rotation compared to the unperturbed case,

$$\{\delta E, \Delta\theta \text{ from } \Delta E\} \Rightarrow \Delta\phi$$

Benchmarking efforts using existing analytical formalism are on-going (cf. V. Ptitsyn’s presentation).

In conclusion to this “SR” reminder :

- We will resort to numerical stepwise ray-tracing (beyond just the linear model which provides averages, $\overline{\Delta E}$, σ_E , σ_l , etc.), in order to :

- accurately monitor

beam dynamics

polarization dynamics

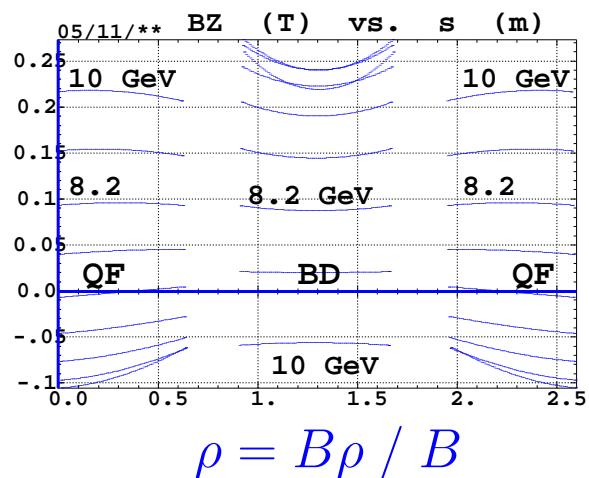
in presence of synchrotron radiation

- push simulations beyond : errors, depolarizing bands, etc.

- In particular, FFAG-type lattices introduce a series of complications that limit the efficiency of linear optical models (compared to, say, CEBAF-like arc optics), and spin dynamics models, as

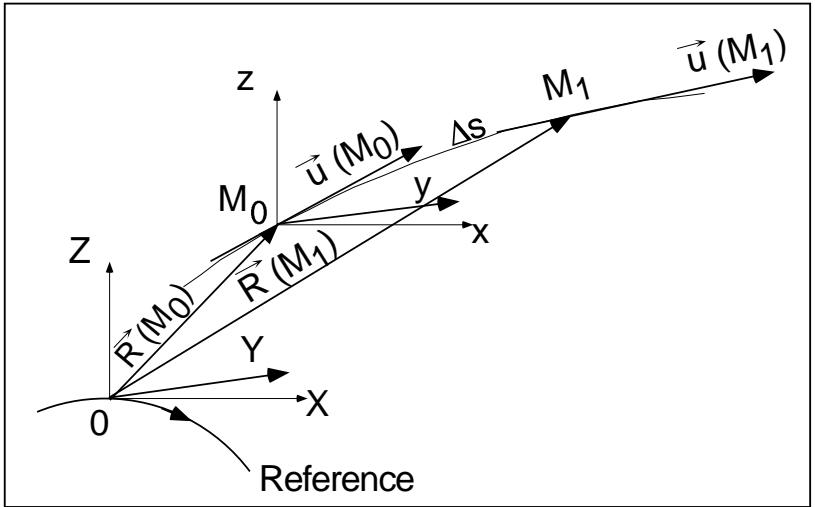
- curvature radii vary all over the lattice cell, and with orbit/energy
- FFAG arcs are not achromatic, so complicating the transport of beam concentration ellipses.

Magnetic field along eRHIC FFAG cell magnets :



2 The Zgoubi numerical machinery

2.1 Particle and spin motion



Position and velocity of a particle, pushed from location M_0 to location M_1 in Zgoubi frame.

- Both equations are solved using a truncated Taylor series in the step size Δs ,

$$\vec{a}(M_1) \approx \vec{a}(M_0) + \frac{d\vec{a}}{ds}(M_0) \Delta s + \dots + \frac{d^n \vec{a}}{ds^n}(M_0) \frac{\Delta s^n}{n!} \quad (1)$$

- Solving particle motion : \vec{a} stands for position \vec{R} or normalized velocity $\vec{u} = \vec{v}/v$,
- Solving spin motion : \vec{a} stands for the spin \vec{S} .

- The local magnetic field and its derivatives fully determine the coefficients $a^{(n)} = d^n a / ds^n$ in this Taylor series.

$$\frac{d(m\vec{v})}{dt} = q(\vec{e} + \vec{v} \times \vec{b}),$$

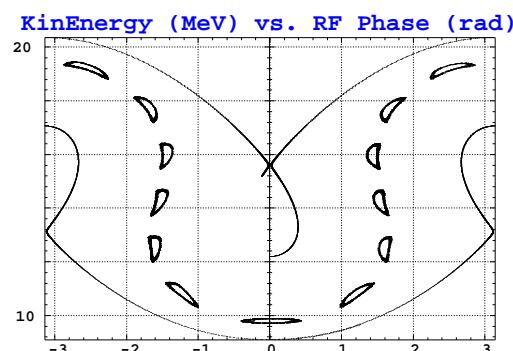
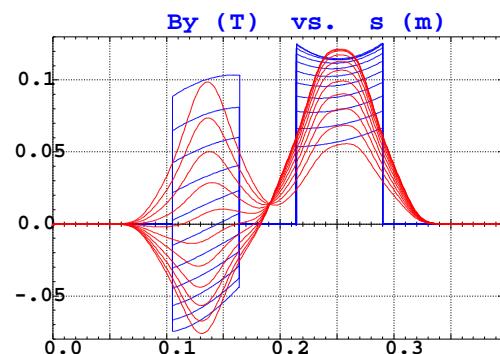
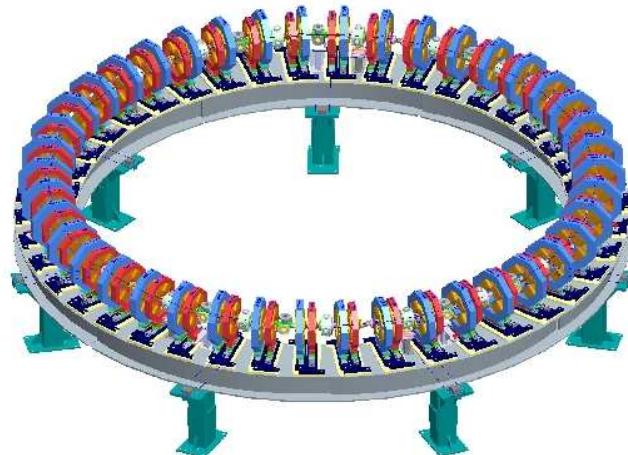
$$\frac{d\vec{S}}{dt} = \frac{q}{m} \vec{S} \times \vec{\omega}$$

$$\text{with } \vec{\omega} = (1 + \gamma G)\vec{b} + G(1 - \gamma)\vec{b}_{\parallel}$$

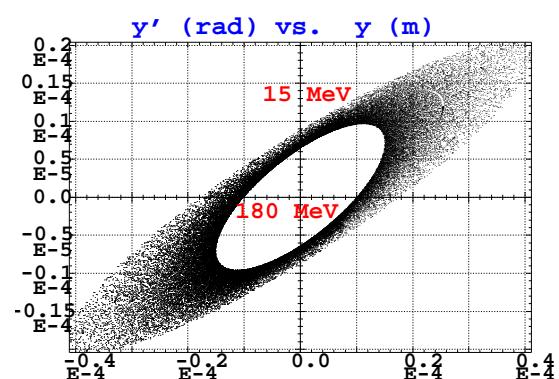
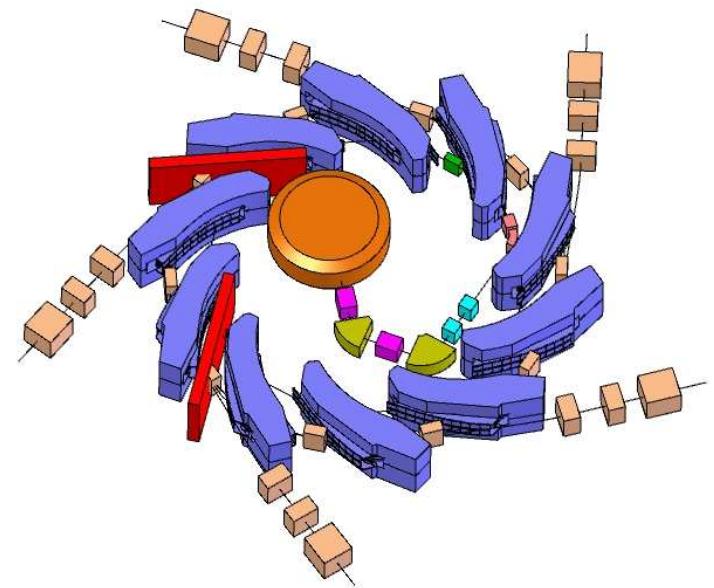
G : gyromagnetic factor, γ : Lorentz relativistic factor, c : velocity of light, q : charge, m : mass.

FFAG R&D using Zgoubi : Zgoubi has tracked in all possible FFAG lattices, for more than a decade, in all sorts countries (EU, Japan, USA...).

- Zgoubi is used as the on-line model at Daresbury Lab's EMMA linear FFAG prototype experiment.



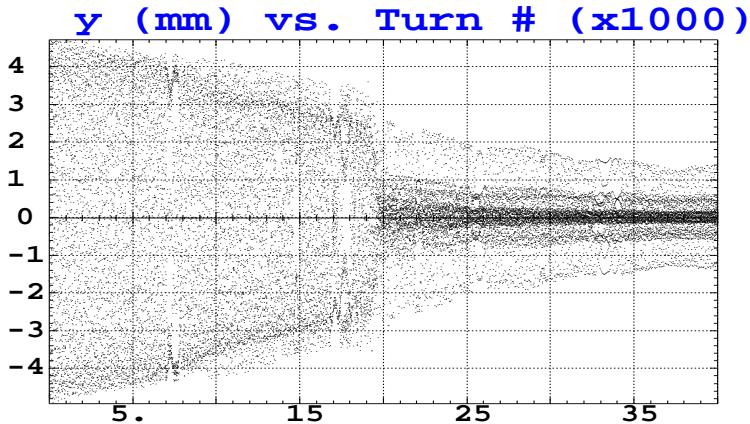
- A multiple extraction protontherapy scaling FFAG, designed using Zgoubi.



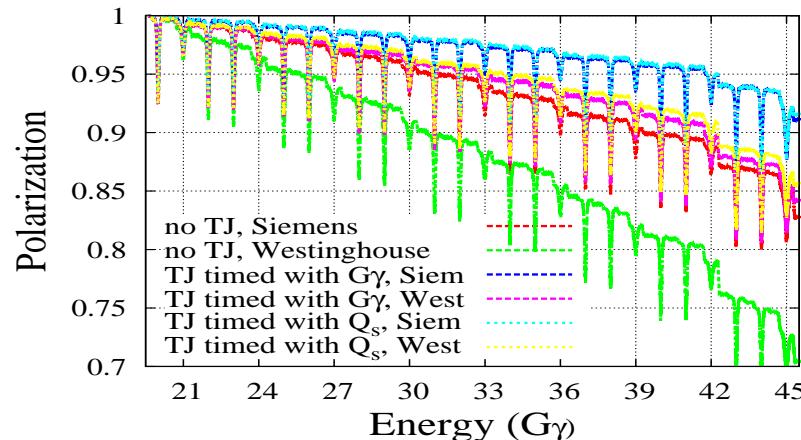
Vertical phase-space,
adiabatic damping over 15000-turn
 $15 \rightarrow 180$ MeV acceleration.

Polarization R&D using Zgoubi

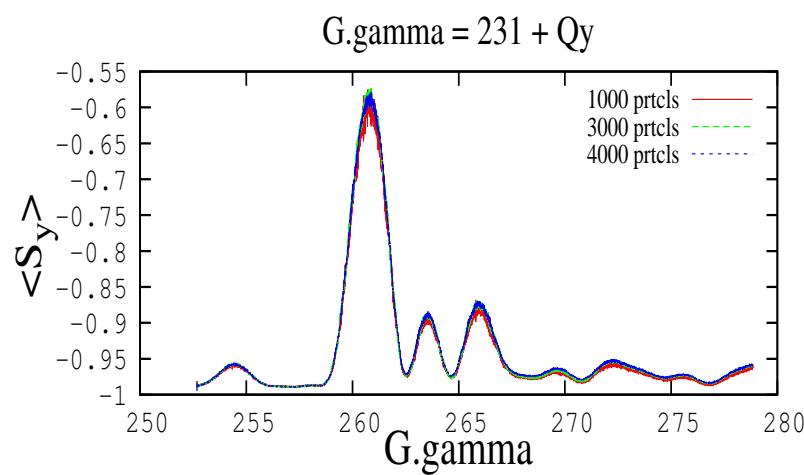
- Since the late 1980s at SATURNE, the second polarized proton machine.
- Nowadays an on-line model in the AGS, using 3-D OPERA field maps of the helical snakes



Horizontal excursion from injection to transition energy in AGS.



Evolution of the polarization along AGS acceleration cycle, 10^3 particles, 6-D bunch [Y. Dutheil].



Evolution of the polarization during snake resonance crossing. 4000 particles, 100,000 turns.

- Used since 2010 to study polarization transport in RHIC

2.2 Synchrotron radiation in Zgoubi

1/ The probability of emission of a photon follows a Poisson law

$$p(k) = \frac{\Lambda^k}{k!} e^{-\Lambda} \quad \text{with} \quad \Lambda = \langle k \rangle, \Lambda = \langle (\Delta k)^2 \rangle$$

since the number of photons, k , radiated within an integration step Δs is a few units or so :

- For instance, over a step Δs , a 10 GeV electron will radiate, on average,

$$\Lambda = \frac{129.5 E[\text{GeV}]}{2\pi} \frac{\Delta s}{\rho} \approx 206 \times \frac{\Delta s}{\rho}, \text{ i.e., about 0.02 photons, if one takes } \begin{cases} \text{a step } \Delta s = 1 \text{ cm} \\ \text{and } 100 \text{ m local radius} \end{cases}$$

- At each integration step in Zgoubi, k is drawn from $p(k)$ using a rejection method.

2/ These k photons are assigned energies $\epsilon = h\nu$, using the cumulative density function for the energy (tabulated in Zgoubi)

$$\mathcal{P}(\text{photon energy} \in [0, \epsilon]) = \frac{3}{5\pi} \int_0^{\epsilon/\epsilon_c} \int_{\epsilon/\epsilon_c}^{\infty} K_{5/3}(x) dx$$

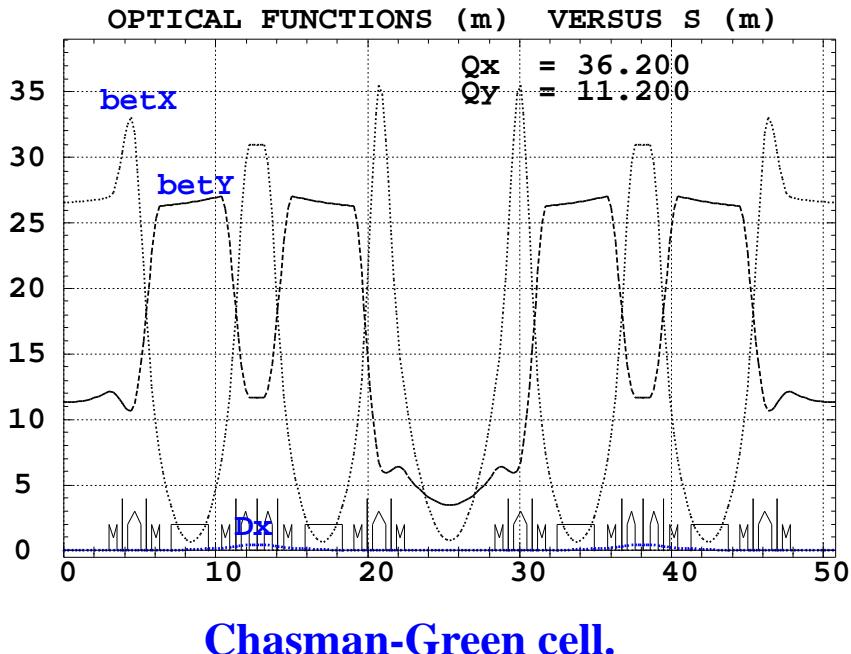
($K_{5/3}$ = modified Bessel function, $\epsilon_c = 3\hbar\gamma^3 c/2\rho$ is the local “critical frequency” of the radiation).

- At each integration step, a random uniform value $0 < \mathcal{P} \leq 1$ yields ϵ/ϵ_c .

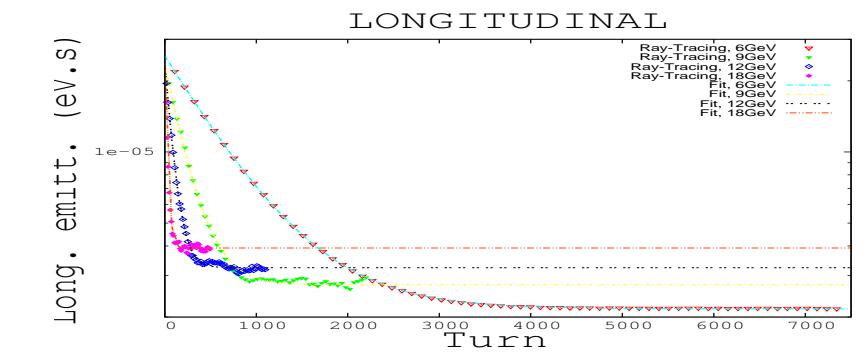
3/ Concluding this process : the particle energy is updated, at each integration step.

Stochastic synchrotron radiation R&D using Zgoubi

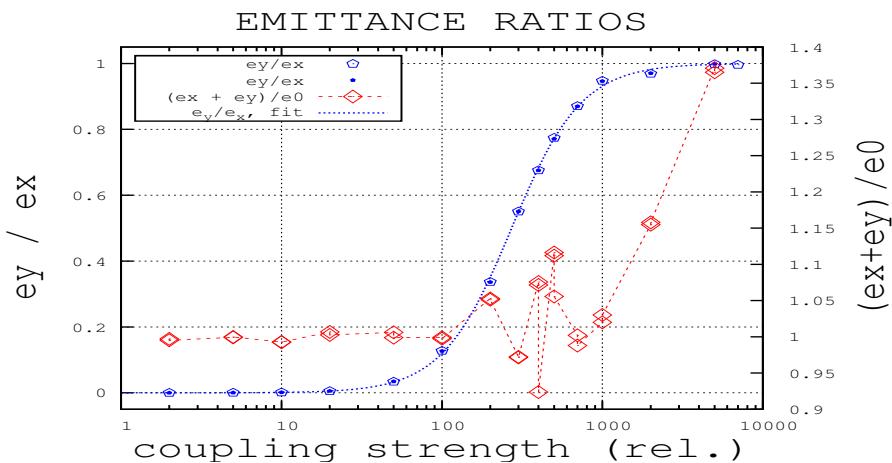
- Installed in the late 1990s to study emittance growth effects at TESLA IP (EU ILC at that time),
- Used in the recent past for polarization studies in the superB project,
- Benchmarked recently (2008 on) for SR effects in rings, using the ESRF lattice :



Benchmarking virtue : provides analytical formulas for damping times, emittance damping, etc.



Longitudinal damping, 6, 9, 12 and 18 GeV.
 2×10^3 particles ray-traced.



Coupled lattice. 6 GeV. 2×10^3 particles, 2×10^4 turn.

2.3 Spin diffusion

... is a spin-off ! of what precedes

comes for free :

$$\left. \begin{array}{c} \text{SPIN DYNAMICS} \\ + \\ \text{STOCHASTIC ENERGY LOSS BY SR} \end{array} \right\} \Rightarrow \text{SPIN DIFFUSION}$$

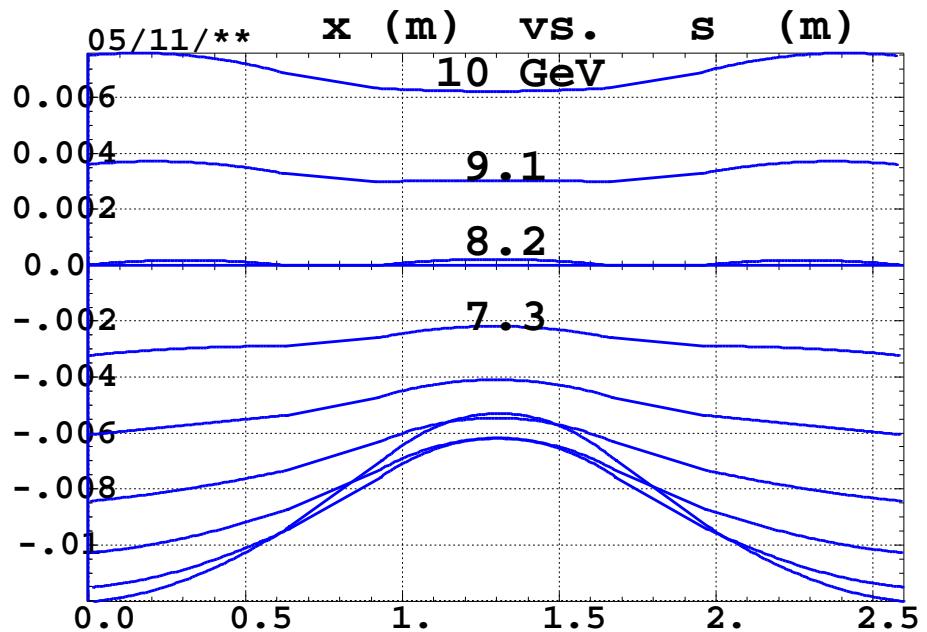
In conclusion to this “Numerical Method” section :

- Zgoubi has tracked problems far more delicate than the FFAG eRHIC ring, in terms of
 - field non-linearities,
 - machine length,
 - number of turns,
 - periodic lattice constraints (e.g., spin, SR),
 - CPU time cost
- End-to-end tracking of 9-D polarized electron bunches radiating along the full FFAG eRHIC installation, should not be much of an issue.
- Expect highest accuracy on particle and spin motion, from accuracy on field models (including field maps wherever necessary)
- Potential for high accuracy on statistics, with large number of particles, with short CPU time given the reduced number of turns.

3 Brief overview of eRHIC FFAG lattice, working hypotheses

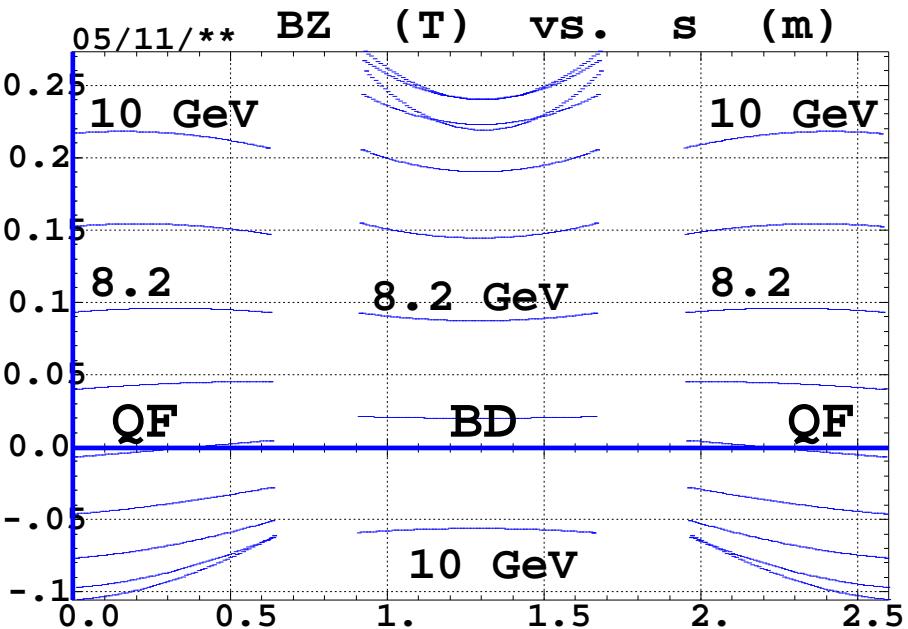
ORBITS :

(the graphic does not account for magnet tilt)



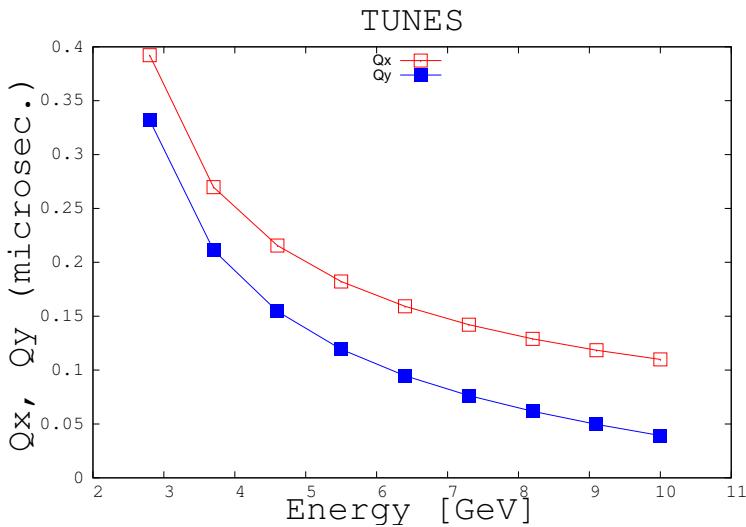
FIELD ALONG ORBITS :

Trajectory curvature varies continuously across the magnets.

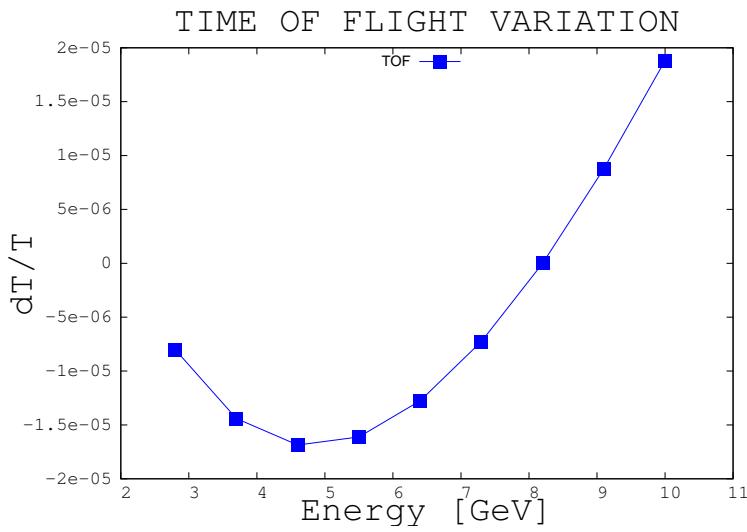


Overview of eRHIC lattice - working hypotheses (cont'd)

TUNES



TIME OF FLIGHT

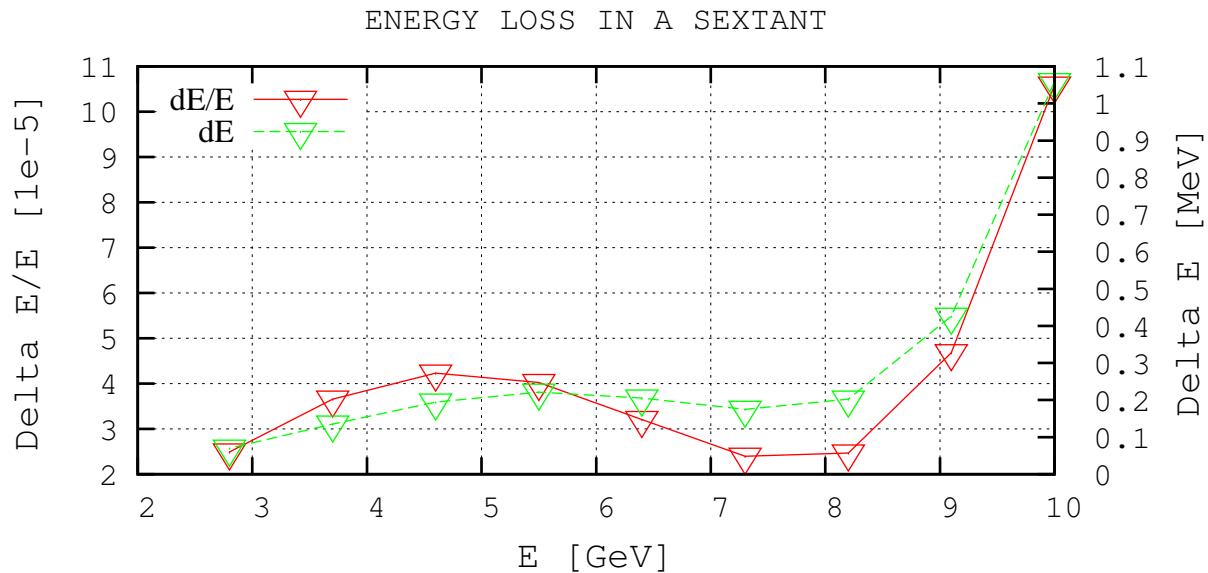


- Note : Tunes and time of flight, together with lattice cell parameters as shown here, have been subject to extensive simulations using various other codes :
MADX, MUON1, SYNCH, etc., all producing consistent results.

4 Zgoubi tracking results, now : dynamical effects of SR in FFAG eRHIC

4.1 Energy loss and spin rotation in a sextant

- Track 10^4 particles through one sextant (138 F-D-F cells), including stochastic SR.
 - All emittances are zero at start of the arc,
 - All starting spins are aligned on longitudinal axis.



- Note : Energy loss in these eRHIC FFAG lattices have been subject to extensive simulations using various other codes : MUON1, Ptitsyn's, etc., all producing consistent results.
- Follows anyway in a cell, as an approximation to $\Delta E = \eta E^2 B^2 \Delta t$, taking $\rho \approx l/\theta$ in a magnet,

$$\overline{\Delta E} [\text{MeV}] = 2 \overline{\Delta E_{QF}} + \overline{\Delta E_{BD}} \approx 0.96 \times 10^{-15} \gamma^4 \left(2 \frac{\Delta\theta_{QF}}{|\rho_{QF}|} + \frac{\Delta\theta_{BD}}{|\rho_{BD}|} \right) \quad (2)$$

- Energy loss, spin rotation : comparison between Zgoubi tracking and analytical models.

Beam E (GeV)	QF, BD (m)	Trajectory deflection		Av. energy loss		Av. spin angle [360]	
		in QF, BD		in an arc		in an arc	
		$\Delta\theta_{QF}$, $\Delta\theta_{BD}$	(mrad)	Zgoubi (keV)	Equ. 3 (keV)	Zgoubi (deg)	Equ. 4 (deg)
2.8	105.20165, -40.239231	5.9410, -18.6385		68.832	69.84	20.477	20.437
5.5	476.85135, -79.974777	1.3107, -9.37796		218.26	221.15	53.146	53.001
7.3	-564.59460, -165.10310	-1.1070, -4.54262		173.44	174.85	165.08	165.290
8.2	-289.04224, -308.39349	-2.1623, -2.43196		200.69	202.33	85.804	85.565
10	-155.28610, 580.01349	-4.0248, 1.29307		1050.36	1058.8	132.37	132.726

- Approximation to $\Delta E \propto E^2 B^2 \Delta t$:

- taking an average radius $\rho_{QF} = l_{QF}/\Delta\theta_{QF}$ in QF,
- and $\rho_{BD} = l_{BD}/\Delta\theta_{BD}$ in BD,

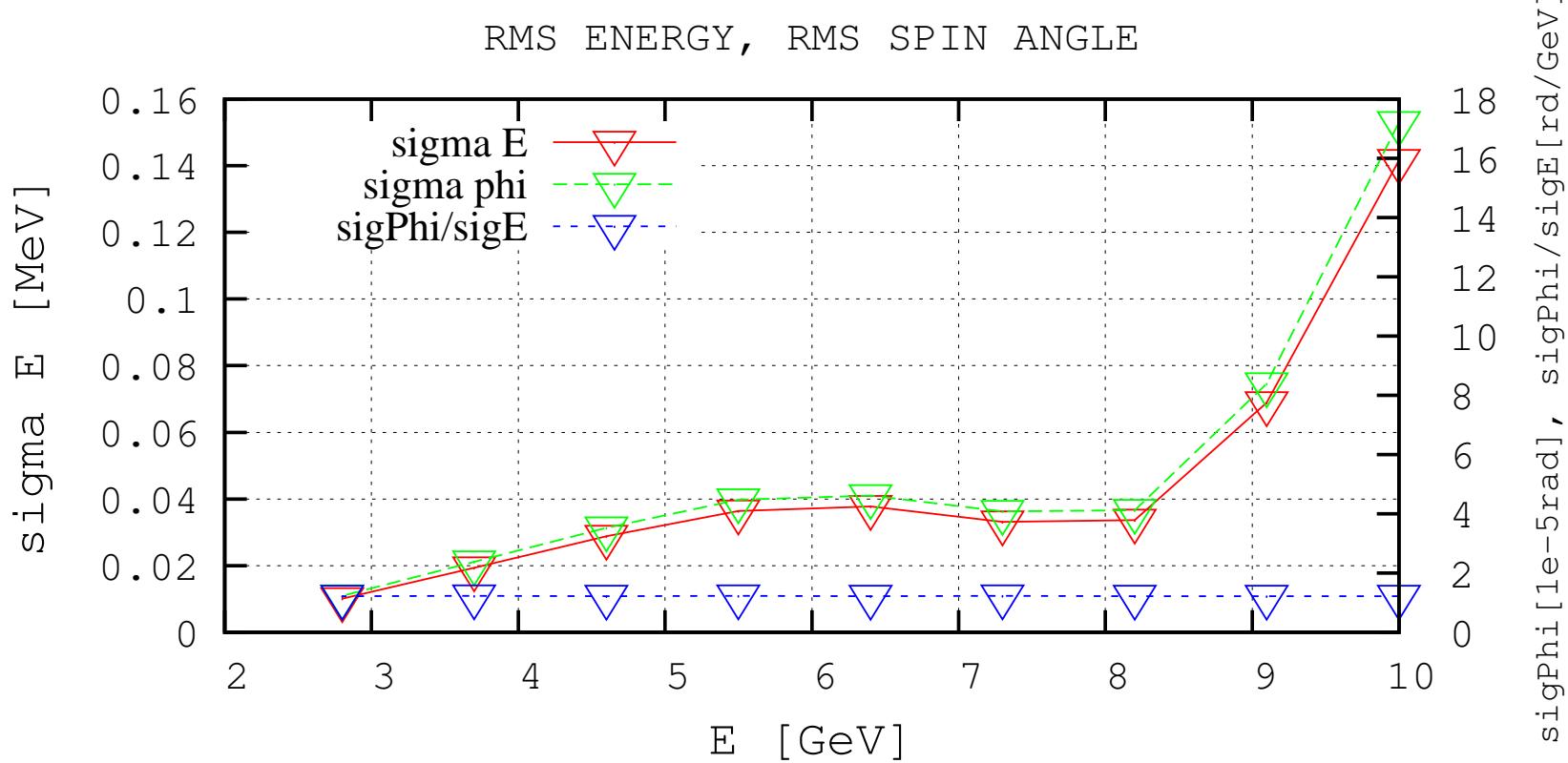
$$\overline{\Delta E}[MeV] = 2 \overline{\Delta E_{QF}} + \overline{\Delta E_{BD}} \approx 0.96 \times 10^{-15} \gamma^4 \left(2 \frac{\Delta\theta_{QF}}{|\rho_{QF}|} + \frac{\Delta\theta_{BD}}{|\rho_{BD}|} \right) \text{ per cell} \quad (3)$$

- The expected spin rotation, rightmost column, is proportional to γ and to particle deflection angle :

$$\phi = G\gamma\theta = G\gamma(2\Delta\theta_{QF} + \Delta\theta_{BD}) \text{ per cell} \xrightarrow{\times 138} \text{per arc} \quad (4)$$

4.2 Energy spreading, spin motion spreading

- Track 10^4 particles through one sextant (138 F-D-F cells). Compute distribution sigmas.
 - All emittances zero at start of the arc.
 - All spins aligned on longitudinal axis at start.



- Energy and spin angle spreading, comparison between
 - E-spread from Zgoubi tracking
 - and quadratic summation of magnet contributions, equation 5 below.

	Radius in QF, BD	Deflect. angle	$\sigma_E, 1 \text{ arc}$	$\sigma_E, 1 \text{ arc}$	σ_ϕ	σ_ϕ/σ_E
Energy (GeV)	$l_F/\Delta\theta_{QF}, l_D/\Delta\theta_{BD}$ (m)	$\Delta\theta_{QF}, \Delta\theta_{BD}$ (mrad)	Zgoubi (keV)	Equ. 5 (keV)	[10^{-5} rad]	[rad/GeV]
2.8	105.201, -40.2392	5.9409, -18.638	10.06	9.85	1.23801	1.231
5.5	476.851, -79.9747	1.3106, -9.3779	36.42	35.84	4.67312	1.283
7.3	-564.594, -165.103	-1.1069, -4.5426	33.16	33.09	4.09175	1.234
8.2	-289.042, -308.393	-2.1623, -2.4319	33.63	33.17	4.12160	1.226
10	-155.286, 580.013	-4.0248, 1.2930	141.57	138.84	17.2150	1.216

- Assuming $\langle (1/\rho)^2 \rangle \approx 1 / \langle \rho^2 \rangle$, and neglecting cell-to-cell mismatch, yields

$$\sigma_E \approx \sqrt{2\sigma_{E,QF}^2 + \sigma_{E,BD}^2} \approx 1.94 \times 10^{-14} \gamma^{7/2} \sqrt{2 \frac{\Delta\theta_{QF}}{\rho_{QF}^2} + \frac{\Delta\theta_{BD}}{\rho_{BD}^2}} \quad (5)$$

supporting tracking data, and providing quick estimate.

4.3 End-to-end tracking simulation

A ring is built from 6 arcs (no drifts) and includes one, zero-length, 0.9 GeV longitudinal kick simulation.

Working hypotheses :

- (i) 10000 electrons are tracked, over 9 eRHIC ring turns, up to 10 GeV
- (ii) At the beginning of all eRHIC turn, the beam centroid is centered on the FFAG orbit
- (iii) After a complete eRHIC turn (i.e., 6 arcs), the beam is given a longitudinal boost, a point transform that simulates the linac
- (iv) The beam is then tracked through the next arc
- (v) Then, repeat (ii)-(iv), up to top energy, 10 GeV

• Energy spread

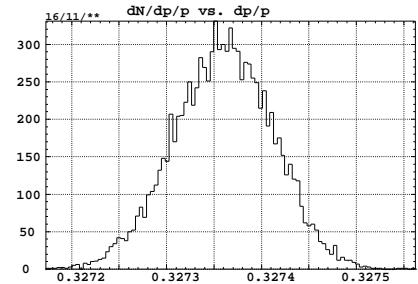
The table below compares

- the sigma values obtained from end-to-end tracking (4th column) :

- to the expected cumulative effect extrapolated in a convolution hypothesis (3rd column), given the data from one arc tracking, with for each arc zero starting 6-D emittance (col. 2),

Beam E (GeV)	σ_E from tracking 1 arc (keV)	Cumulated, 6 arcs, $\sqrt{6 \sum \sigma_E^2}$ (keV)	σ_E from tracking, End-to-end (keV)	Relative difference (10^{-3})
2.8	10.06	24.65	24.80	-6.0
3.7	19.32	53.36	53.24	+2.2
4.6	28.86	88.57	88.78	-2.3
5.5	36.43	125.73	125.99	-2.0
6.4	37.77	156.10	157.47	-8.7
7.3	33.16	175.97	177.32	-7.6
8.2	33.63	194.30	195.64	-6.8
9.1	68.88	257.34	255.52	+7.1
10	141.6	431.82	425.53	+14.1

- What we monitor, here :



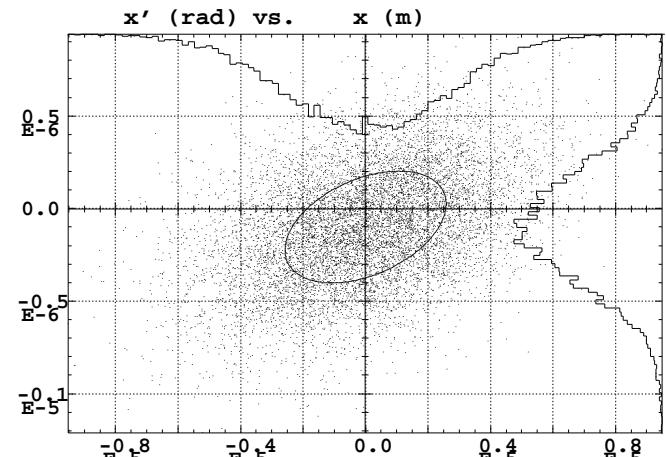
- Differences, rightmost column, are within error bars of both col. 2 and col. 4 tracking data.

• Horizontal emittance growth

- The starting emittance is taken zero.
- The central two columns in the table show the beam centroid drift : **the spiraling is negligible**.
- The right column gives the value of the **beam area in horizontal phase-space** at the end of each turn in eRHIC ring : **the emittance growth due to SR is negligible**.

Energy (GeV)	Beam centroid position, \bar{x}		Cumulated beam area σ_x (m.rad)
	start of turn	end of turn	
2.8	-1.202239	-1.202244	2.87×10^{-15}
3.7			4.44×10^{-15}
4.6			7.24×10^{-15}
5.5			1.44×10^{-14}
6.4	-0.6060720	-0.6057128	2.42×10^{-14}
7.3			3.77×10^{-14}
8.2			5.54×10^{-14}
9.1			1.91×10^{-13}
10	+0.7493935	0.7465509	7.08×10^{-13}

- What we monitor, here :

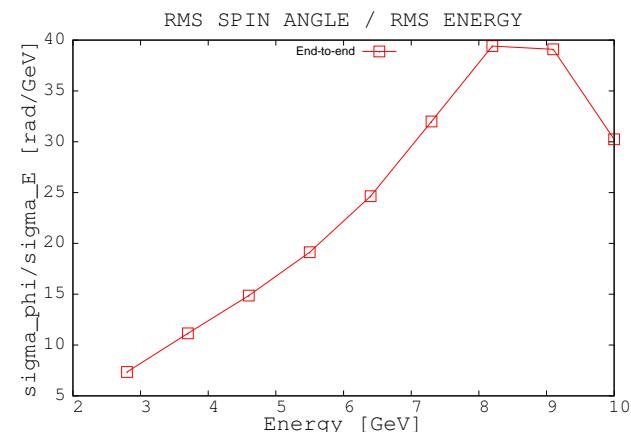
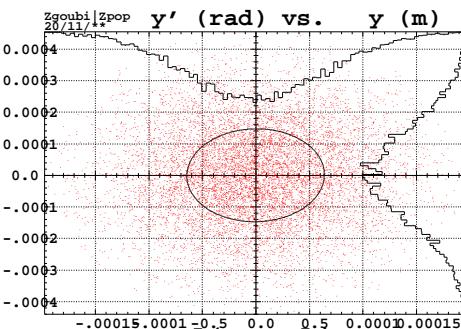


• Spin motion : rotation and diffusion

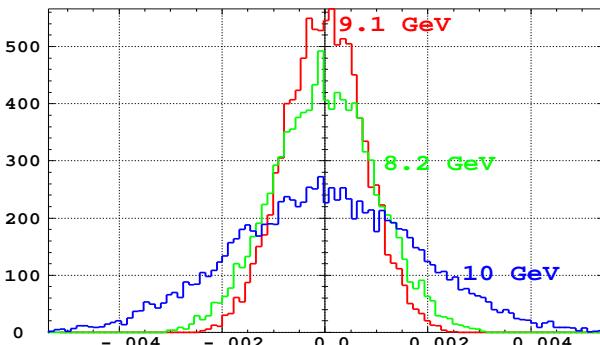
- All spins are along the longitudinal axis at start.
- The second column in the table gives the value of the spin rotation angle at the end of each turn in eRHIC
- The third column gives the rms value of the angle distribution.
- The rightmost two columns show the spreading of the vertical spin component in presence of non-zero vertical motion.

Beam E (GeV)	Case $\epsilon_x = \epsilon_y = 0$		$\epsilon_y = 50 \text{ mm.mrd}$		
	Spin angle		$\epsilon_x = 0$		$\epsilon_x = 50$
	ϕ [360] (deg)	σ_ϕ (deg)	σ_ϕ (deg)	σ_{S_y} (10^{-3} deg)	σ_{S_y} (10^{-3} deg)
2.8	122.99	0.01044	0.01023	1.5577	1.5704
3.7	48.25	0.03401	0.03330	0.7815	0.7786
4.6	153.45	0.07562	0.07474	1.8655	1.8819
5.5	167.55	0.13812	0.13736	1.6282	1.6443
6.4	165.18	0.22232	0.22092	0.8827	0.8790
7.3	71.76	0.32504	0.32269	1.2610	1.2674
8.2	87.78	0.44175	0.43942	0.9806	0.9859
9.1	45.99	0.57240	0.57179	0.7638	0.7671
10	111.25	0.73722	0.74996	1.7600	1.7742

- Vertical phase-space at end of turn 9, starting emittance is 10^{-8} rms, cut-off 3σ , final is $9.45 \cdot 10^{-9}$ rms.



- Vertical spin component, end of turns 7, 8, 9, case $\epsilon_y = 50 \text{ mm.mrd}$, $\epsilon_x \approx \text{any}$.



• Some injection errors

- All spins are along the longitudinal axis at start.
- Starting emittances are all zero.
- Injection mis-positionning is introduced,
 either $\delta x_0 = 1 \text{ mm}$
 or $\delta y_0 = 1 \text{ mm}$.

Beam E (GeV)	$\epsilon_x = \epsilon_y = 0$					
	Spin angle ϕ [360] (deg)	σ_ϕ	$\delta x_0 = 1 \text{ mm}$		$\delta y_0 = 1 \text{ mm}$	
			ϕ [360] (deg)	σ_ϕ	σ_ϕ (deg)	σ_{S_y} (10^{-3} deg)
2.8	122.99	0.01044	122.93	0.01116	122.99	0.01084
3.7	48.25	0.03401	48.31	0.33803	48.17	0.03387
4.6	153.45	0.07562	153.51	0.75093	153.20	0.07366
5.5	167.55	0.13812	167.49	0.13728	168.04	0.13495
6.4	165.18	0.22232	165.23	0.22064	164.26	0.21821
7.3	71.76	0.32504	71.83	0.32300	70.27	0.32082
8.2	87.78	0.44175	87.73	0.44012	90.081	0.44413
9.1	45.99	0.57240	46.05	0.56982	42.58	0.57895
10	111.25	0.73722	111.30	0.73401	106.34	0.76539

$\overbrace{\hspace{10em}}$
Not much effect on angle
neither on spreading

5 In conclusion

Up to 10 GeV, stochastic SR :

- has a very limited effect on beam dynamics : transverse emittances, energy spread, bunch lengthening,
- has a very limited effect on spin decoherence : angle diffusion is less than a degree after 9 passes from injection to top energy.
- Linear beam dynamics model on the one hand, classical spin dynamics model on the other hand, are still a good approach.

6 Plans - in more or less chronological order

- Ramp down in energy, get beam averages and sigmas down to dump energy
- Do continuous energy scan of the arcs, for search of forbidden/depolarizing energy bands.
Develop and make that tool part of the optics design toolbox
- Add Matching sections to the arcs, linac, etc.
Complete the full eRHIC complex from injection.
- End-to-end tracking of 9-dimensional polarized bunch with realistic starting phase-space and spin distributions.
Lattice studies, beam and polarization dynamics studies.

DONE - THANK YOU

A End to end tracking

What is simulated here :

(i) a 10,000 particle bunch is launched at entrance to arc#1 of the ring, on 2.8 GeV orbit with all initial densities Dirac.

(ii) it is tracked, including spin, over a full RHIC turn, six arcs, no cavities, in presence of SR loss

(iii) at the end of the turn the bunch is, 1/ given a 0.9 GeV kick, 2/ centered on the next orbit at entrance to arc#1 of the ring, ready for a new turn.

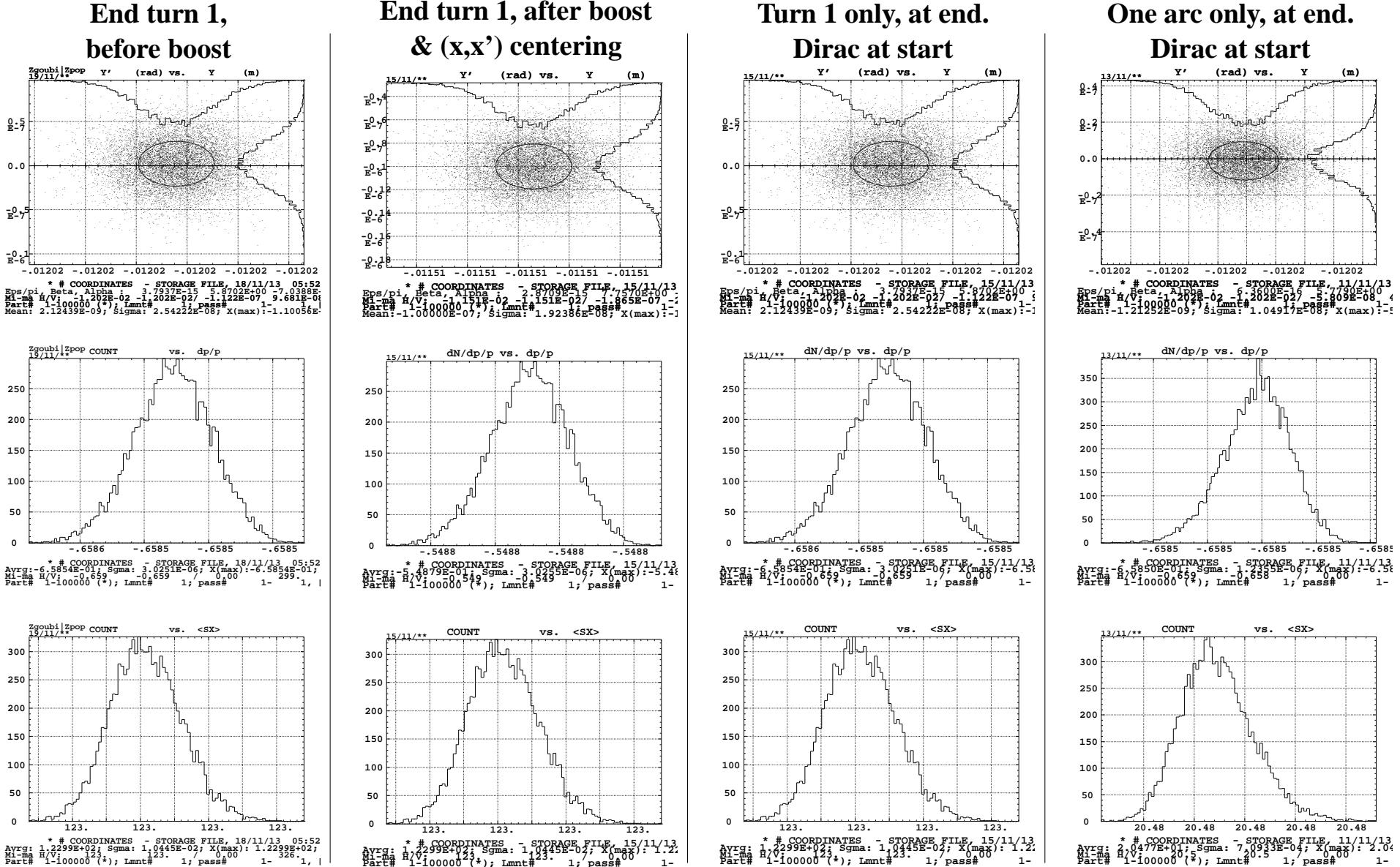
Hence the bunch will start that new turn with the densities (horizontal extent, energy and spin) as acquired during the previous ones.

(iv) the sequence (ii)-(iii) is done 9 times, then the bunch has reached ≈ 10 GeV and done a (*full*) last RHIC turn with that energy.

The next plots display the radial, energy, and spin distributions, after turns # 1 (2.8 GeV) and 9 (10 GeV).

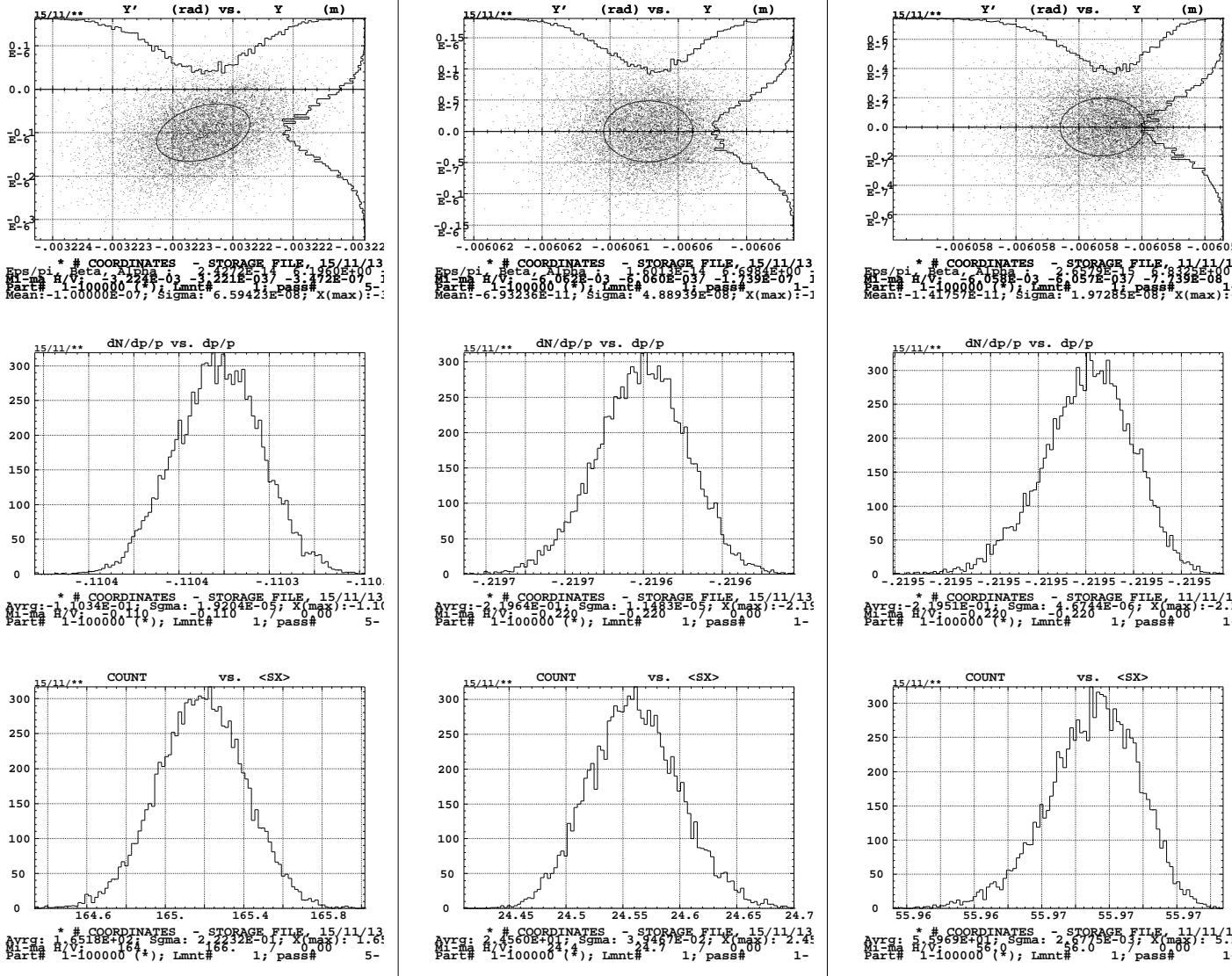
- End-to-end tracking, first eRHIC turn of 9. At start of arc#1 conditions for the 10,000 particles are : $x = x_{\text{orbit}}$, $\epsilon_x = 0$, $y = 0$, $\epsilon_y = 0$, $E = 1.9 \text{ GeV}$, $\epsilon_l = 0$,

End of eRHIC turn#1, (x,x') space, dp/p and spin angle densities. First column is end-to-end before 0.9 GeV boost, second column is end-to-end right after 0.9 GeV boost and beam centering to FFAG orbit#2, third column is just turn#1 with zero initial, right column is just an arc with zero initial.



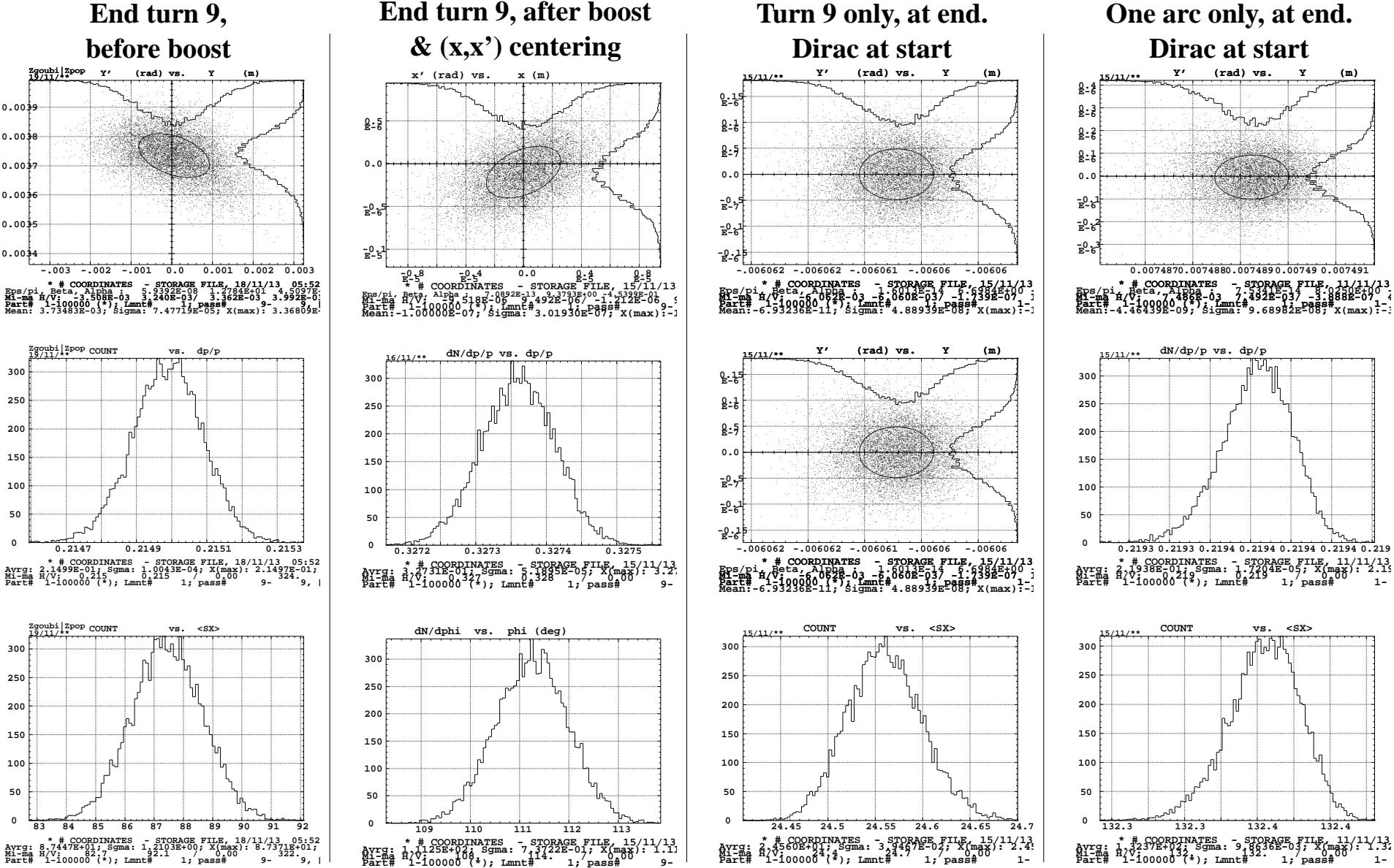
- End-to-end tracking, 5th eRHIC turn of 9. At start of arc#5 conditions for the 10,000 particles are from end of previous arc yet with beam re-centered on new orbit, and with 0.9 GeV boost.

End of eRHIC turn#5, (x,x') space, dp/p and spin angle densities.



- End-to-end tracking, last turn (9th). At start of arc#9 conditions for the 10,000 particles are from end of previous arc yet with beam re-centered on new orbit, and with 0.9 GeV boost.

End of eRHIC turn#9, (x,x') space, dp/p and spin angle densities.



B Particle and spin motion in Zgoubi

- The Lorentz equation governs the motion of a particle of charge q , relativistic mass m and velocity \vec{v} in electric and magnetic fields \vec{e} and \vec{b} .
- The Thomas-BMT equation governs spin motion, write respectively (in Zgoubi reference frame as defined in Fig. 1)

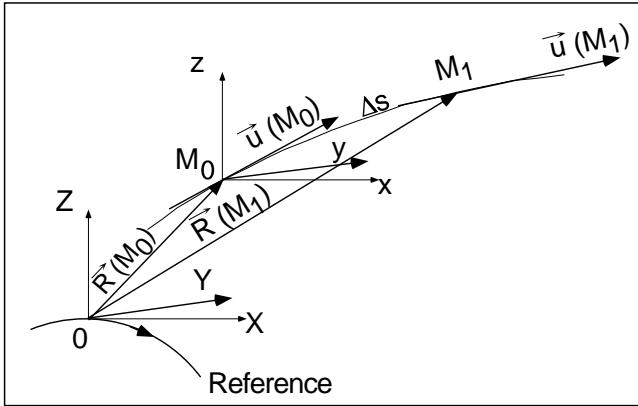


Figure 1: Position and velocity of a particle, pushed from location M_0 to location M_1 in Zgoubi frame.

$$\frac{d(m\vec{v})}{dt} = q(\vec{e} + \vec{v} \times \vec{b}), \quad \frac{d\vec{S}}{dt} = \frac{q}{m} \vec{S} \times \vec{\omega} \quad (6)$$

One can introduce

$$()' = d() / ds, \vec{u} = \vec{v} / v, ds = v dt, \vec{u}' = d\vec{u} / ds, m\vec{v} = mv\vec{u} = q B\rho \vec{u} \quad (7)$$

with $B\rho$ the rigidity of the particle.

- The equations of motion for the particle and its spin can be rewritten in the handy forms, as found in Zgoubi source code,

$$(B\rho)' \vec{u} + B\rho \vec{u}' = \frac{\vec{e}}{v} + \vec{u} \times \vec{b}, \quad (B\rho) \vec{S}' = \vec{S} \times \vec{\omega} \quad (8)$$

with, for the local spin precession axis,

$$\vec{\omega} = (1 + \gamma G) \vec{b} + G(1 - \gamma) \vec{b}_{\parallel} + \gamma(G + \frac{1}{1 + \gamma}) \frac{\vec{e} \times \vec{v}}{c^2} \quad (9)$$

G the gyromagnetic factor, γ the Lorentz relativistic factor, c the velocity of light. Both equations are solved using a truncated Taylor series in the step size Δs ,

$$\vec{a}(M_1) \approx \vec{a}(M_0) + \vec{a}'(M_0) \Delta s + \dots + \vec{a}^{(n)}(M_0) \frac{\Delta s^n}{n!} \quad (10)$$

- \vec{a} stands for either the position \vec{R} and velocity \vec{u} , or for the spin \vec{S} .

In addition, a scalar form of equation 10 (replace $\vec{a}(M)$ by $a(M)$) will push the rigidity $B\rho(M)$ and time $T(M)$ in the presence of electric fields.

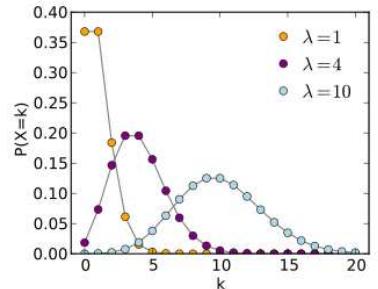
- The coefficients $a^{(n)} = d^n a / ds^n$ in these Taylor series are obtained by recurrent differentiation of equation 8. In a practical manner this means that the magnetic fields and their derivatives, $d^n \vec{e} / ds^n$, $d^n \vec{b} / ds^n$, are needed, they are provided using analytical field models or measured or computed field maps. The Taylor series truncation (eq.10) is beyond derivative order $n = 4$ to $n = 6$, depending upon options. Field derivatives are always computed up to second order, and beyond that in some elements and upon option.

C Synchrotron radiation in Zgoubi

1/ The probability of emission of a photon follows

- a Poisson law if the number of photons, k , radiated within an integration step Δs is a few units or so

$$p(k) = \frac{\Lambda^{-k}}{k!} e^{-\Lambda} \quad \text{with} \quad \Lambda = \langle k \rangle, \Lambda = \langle (\Delta k)^2 \rangle$$



- a Gauss-Laplace law if $\langle k \rangle = \Lambda \gg 1$.

- For instance, a 10 GeV electron will radiate, on average, per trajectory arc element $\Delta\theta$, a number of photons

$$\Lambda = \frac{129.5 E[GeV] \Delta\theta}{2\pi} = \frac{\gamma}{94.916} \Delta\theta \approx 206 \times \Delta\theta \quad \text{i.e., about 2 photons for } \Delta\theta = \frac{10 \text{ cm step}}{10 \text{ m radius}}$$

This justifies using a Poisson law.

- The parameter $\Lambda = \langle k \rangle$ in a step, for the Poisson generator, is given by :

$$\langle k \rangle = \Lambda = \frac{5er_0}{2\hbar\sqrt{3}} \beta^2 B\rho \frac{\Delta s}{\rho} \quad (11)$$

with Δs = step size, $B\rho$ = rigidity, $r_0 = e^2/(4\pi\epsilon_0 m_0 c^2)$ = classical radius of the particle, m_0 = rest-mass, e = elementary charge, $\hbar = h/2\pi$, h = Planck constant, $\beta = v/c$.

- At each integration step, k is drawn from the Poisson pdf using a rejection method.

2/ These k photons are assigned energies $\epsilon = h\nu$,

- This is done using the cumulative density function for the energy, i.e., the probability that a photon has its energy in $]0, \epsilon]$, namely

$$\mathcal{P}(\epsilon/\epsilon_c) = \frac{3}{5\pi} \int_0^{\epsilon/\epsilon_c} \int_{\epsilon/\epsilon_c}^{\infty} K_{5/3}(x) dx$$

- $K_{5/3}$ = modified Bessel function (diverges for $\epsilon \rightarrow 0$),
 - $\epsilon_c = \hbar\omega_c$, $\omega_c = 3\gamma^3 c / 2\rho$, critical frequency of the radiation, is evaluated at the the current step from the current values of γ and $\rho = B\rho/B$
 - In the low frequency region ($\epsilon/\epsilon_c < 10^{-2}$ in Zgoubi), $\mathcal{P}(\epsilon/\epsilon_c)$ is approximated by
- $$\mathcal{P}(\epsilon/\epsilon_c) = \frac{12\sqrt{3}}{2^{1/3} 5 \Gamma(\frac{1}{3})} (\epsilon/\epsilon_c)^{1/3} \quad (\text{precision better than 1\%})$$
- Beyond, 40 values of $\mathcal{P}(\epsilon/\epsilon_c)$ (covering $10^{-2} < \epsilon/\epsilon_c \leq 10$) have been tabulated in Zgoubi, thus linear interpolation.
 - In order to get ϵ/ϵ_c a random value $0 < \mathcal{P} < 1$ is first generated uniformly, then ϵ/ϵ_c is drawn from the cdf.
 - Concluding this process : the particle energy is updated.

References

- [1] Synchrotron radiation effects in transfer lines, A Tkatchenko, G. Leleux, int. rep. LNS/GT/90-10, SATURNE, CEA Saclay, 1990.
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